Warwick Mathematics Exchange

MA133

Differential Equations

2022, June 13th

Desync, aka The Big Ree

Contents

Introduction

Differential equations are equations that relate functions with their derivatives. In this course, we mainly work with functions of single variables, with vector-calculus being the focus of MA134 *Geometry & Motion*. This module is very computational in nature, with most of the questions you encounter simply asking you to write, solve and interpret differential equations and recurrence relations. This computational aspect is somewhat shared with MA106 *Linear Algebra*.

This document is intended to broadly cover all the topics within the Differential Equations module. Integration skills from A-Levels are assumed and will not be covered extensively. Knowledge of recurrence relations (from Edexcel FP2/DM2 o.e.) is not assumed.

This document is not designed to be a replacement for lecture notes, although you can certainly use it as one if you already have a solid understanding of the content from outside of the course - much of the content is covered in a different order than is taught in the course (for example, partial derivative notation is covered in the first section of these notes, before it would be covered in lecture notes, in order to keep the notation section comprehensive and cohesive), so it is not recommended to learn the module from these notes unless you are familiar with most of the content already.

Many of the techniques developed here are used extensively in further (and many core!) modules - notably, MA134, MA250, MA261, MA263, and MA269, so take care not to just forget everything you've learned once the exam has passed by.

Some of the techniques used towards the end of this module will perhaps not make much sense until you have completed MA106 and MA134, so don't worry if you're reading this ahead of time and are finding some of the techniques somewhat opaque.

Due to the computational nature of this module, this document mainly consists of a checklist of how to solve different types of differential equations and recurrence relations, with not much in the way of theory.

Disclaimer: This document was made by a first year student with a severe disinterest in calculus. I make *absolutely no guarantee* that this document is complete nor without error. In particular, any content covered exclusively in lectures (if any) will not be recorded here. Additionally, this document was written at the end of the 2022 academic year, so any changes in the course since then may not be accurately reflected.

Notes on formatting

New terminology will be introduced in *italics* when used for the first time. Named theorems will also be introduced in *italics*. Important points will be **bold**. Common mistakes will be <u>underlined</u>. The latter two classifications are under my interpretation. YMMV.

Content not taught in the course will be outlined in the margins like this. Anything outlined like this is not examinable, but has been included as it may be helpful to know alternative methods to solve problems.

The table of contents above, and any inline references are all hyperlinked for your convenience.

Newton's notation will not be used in this document. Lagrange's notation will be preferred, with Leibniz's notation used wherever differentials are more helpful (i.e. separable equations).

History

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Authors

This document was written by R.J. Kit L., a maths student. I am not otherwise affiliated with the university, and cannot help you with related matters.

Please send me a PM on Discord @Desync#6290, a message in the WMX server, or an email to Warwick.Mathematics.Exchange@gmail.com for any corrections. (If this document somehow manages to persist for more than a few years, these contact details might be out of date, depending on the maintainers. Please check the most recently updated version you can find.)

If you found this guide helpful and want to support me, you can buy me a coffee!

(Direct link for if hyperlinks are not supported on your device/reader: ko-fi.com/desync.)

*Storing dates in big-endian format is clearly the superior option, as sorting dates lexicographically will also sort dates chronologically, which is a property that little and middle-endian date formats do not share. See ISO-8601 for more details. This footnote was made by the computer science gang.

1 Functions and Variables

1.1 Terminology & Notation

1.1.1 Variables

Variables measure things. We can classify them into *independent* and *dependent* variables.

If a variable, for example, x, is a function of another variable, say, t, then x would be the dependent variable as its value is dependent on t, the independent variable. Usually, we see this written as $x(t)$.

There doesn't have to be a one-to-one correspondence between dependent and independent variables either: for example, you could have temperature as a function of position in 3D, $f(x,y,z)$, where f is the dependent variable, and x,y, and z are independent variables.

Dependent variables can usually be differentiated with respect to the independent variable(s).

1.1.2 Derivative Notation

When there is only one independent variable, we may save space and use Lagrange's (prime) notation over Leibniz's (quotient) notation:

$$\frac{dy}{dx} = y'$$
$$\frac{d^2y}{dx^2} = y''$$
$$\frac{d^ny}{dx^n} = y^{(n)}(x)$$

For nth derivatives in Lagrange's notation, do not omit the independent variable as to avoid confusion with exponents.

If the independent variable is time, we may also use Newton's (dot) notation:

$$\frac{dx}{dt} = \dot{x}$$
$$\frac{d^2x}{dt^2} = \ddot{x}$$

Newton's notation becomes rather unwieldy for derivatives of order higher than 2 or 3.

The partial derivative of a function $f(x,y,z)$ with respect to x, is variously written as,

$$\frac{\partial f}{\partial x}, f_x, \partial_x f$$

Other notations exist, but these are the main ones used in MA133 and MA134.

The second-order partial derivative of f with respect to x is written as,

$$\frac{\partial^2 f}{\partial x^2}, f_{xx}, \partial_{xx} f, \partial_x^2 f$$

and the second-order mixed derivative of f with respect to x, then y is given by,

$$\frac{\partial^2 f}{\partial y \partial x}, f_{xy}, \partial_{yx} f, \partial_y \partial_x f$$

1.1.3 Properties of Differential Equations

If a differential equation only has one independent variable, it is referred to as an *ordinary differential equation*, or *ODE*. Differential equations involving several independent variables are *partial differential equations.*

The *order* of a differential equation is the order of highest derivative present in the equation.

A differential equation is;

- *autonomous* if the independent variable does not appear in the ODE
- *linear* if the ODE can be written in the form, $a(t)x + b(t)x' + c(t)x'' + \cdots = f(t)$.
- *homogeneous* if $f(t) = 0$ in the expression above.

1.2 Existence and Uniqueness of Solutions

Consider the ODE,

$$x'(t) = f(x,t)$$

If both $f(x,t)$ and $\frac{\partial f}{\partial x}$ exist and are continuous for $x \in (a,b)$ and $t \in (c,d)$, then, for any $X \in (a,b)$ and $T \in (c,d)$, the ODE has a unique solution on some open interval containing T (You will learn the formal definition of continuity in MA131 *Analysis II*).

Split up more finely, the theorem says that, if $\frac{dx}{dt} = f(x,t)$ and $x(a) = b$, then, a solution exists if $f(x,t)$ is continuous near (a,b), and that the solution is unique if $\frac{\partial f}{\partial x}$ is continuous near (a,b).

Example.

$$x^2 + t^2 \frac{dx}{dt} = 0; \quad x(0) = c; \quad c \neq 0$$

At $t = 0$, the equation reduces to $x^2 = 0$, but we have $x(0) \neq 0$, so this differential equation does not have any solutions.

Example.

$$\frac{dx}{dt} = \sqrt{x}; \quad x(0) = 0$$

Clearly, the constant function $x(t) = 0$ is a solution, but we also have,

$$x(t) = \begin{cases} 0 & \text{if } t \leq c \\ \frac{(t-c)^2}{4} & \text{if } t > c \end{cases}; \quad c > 0$$

valid for any positive c. So, this differential equation does not have a unique solution.

But it might not be easy to find multiple solutions, so we can check using the theorem above. This differential equation fails the requirements because $x'(0)$ is not well defined, and is hence not continuous.

1.3 Fundamental Theorem of Calculus

Suppose $f : [a,b] \to \mathbb{R}$ is continuous. Let $G(x) = \int_a^x f(z)\,dz$. Then, $\frac{d}{dx}G(x) = f(x)$ (i.e., G is an antiderivative of f) and furthermore, $\int_a^b f(x) = F(a) - F(b)$ for any F such that $F'(x) = f(x)$.

2 First Order Differential Equations

2.1 Linear

For brevity, I will not notate the independent variable from this point onwards (so when you see x or x', I mean $x(t)$ and $\frac{dx}{dt}$, etc.) unless relevant or helpful to the method (i.e., separable equations).

2.1.1 Homogeneous with Constant Coefficients

$$x' + ax = 0$$

If you have a coefficient on x', divide everything by that coefficient to get it into the form above before proceeding.

The solution is given by,

$$x = Ae^{-at}, A = x(0)$$

2.1.2 Separable

$$\frac{dx}{dt} = f(x)g(t)$$

$$\begin{aligned}
\frac{dx}{dt} &= f(x)g(t) \\
\frac{1}{f(x)}\frac{dx}{dt} &= g(t) \\
\int \frac{1}{f(x)}\frac{dx}{dt}\,dt &= \int g(t)\,dt \\
\int \frac{1}{f(x)}\,dx &= \int g(t)\,dt
\end{aligned}$$

After evaluating these integrals, simply rearrange for x.

2.1.3 Homogeneous with Non-Constant Coefficients

$$x' + f(t)x = 0$$

This is just a separable equation:

$$\begin{aligned}
x' &= f(t)x \\
\frac{dx}{dt} &= f(t)x \\
\frac{1}{x}\frac{dx}{dt} &= f(t) \\
\ln x &= \int f(t)\,dt \\
\ln x &= F(t) + C \\
x &= Ae^{F(t)}
\end{aligned}$$

2.1.4 Non-Homogeneous

$$x' + f(t)x = g(t)$$

First, solve the homogeneous version, $x' + f(t)x = 0$, to get the *complementary function*.

Next, we need to get the *particular integral*.

We need to multiply both sides by some function, $I(t)$, such that we can apply the product rule in reverse on the LHS;

i.e., we want,

$$I(t)x' + I(t)f(t)x = (I(t)x)' \tag{1}$$

but,

$$(I(t)x)' = I(t)x' + I(t)x \tag{2}$$

so by equating (1) and (2), we have $I'(t) = I(t)f(t)$, so $I(t) = e^{\int f(t)\,dt}$.

Now, we have,

$$\begin{aligned} I(t)x' + I(t)f(t)x &= I(t)g(t) \\ (I(t)x)' &= I(t)g(t) \\ I(t)x &= \int I(t)g(t)\,dt \\ x &= \frac{1}{I(t)}\int I(t)g(t)\,dt \end{aligned}$$

Adding this to the complementary function found earlier gives the *general solution*.

$I(t)$ is an *integrating factor* of the differential equation.

2.2 Substitutions for Non-Linear ODEs

2.2.1 Type I

$$x' = f\left(\frac{x}{t}\right)$$

Let $u = \frac{x}{t}$. Then, $x = tu$

$$\begin{aligned} x &= tu \\ x' &= (tu)' \end{aligned}$$

We use the product rule ("*left dee-right plus right dee-left*") here, remembering that the derivative of t with respect to t is 1.

$$\begin{aligned} x' &= tu' + u \\ f(u) &= tu' + u \\ f(u) - u &= tu' \end{aligned}$$

which is a separable differential equation.

2.2.2 Type II

$$x' + f(t)x = g(t)x^n$$

Let $u = x^{1-n}$, so,

$$\begin{aligned} u' &= (1-n)x^{-n}x' \\ u' &= (1-n)x^{-n}\left(g(t)x^n - f(t)x\right) \\ u' &= (1-n)\left(g(t) - f(t)x^{1-n}\right) \\ u' &= (1-n)\left(g(t) - f(t)u\right) \\ u' + (1-n)f(x)u &= (1-n)g(t) \end{aligned}$$

which allows the use of an integrating factor.

2.3 Phase Lines

A non-linear ODE will often not have an explicit solution, but we can still analyse them in a couple of ways. We can identify and classify *fixed points* of an autonomous ODE with *phase lines*.

Given an ODE,

$$x' = f(x)$$

draw a graph with x' on the vertical axis, and x on the horizontal axis.

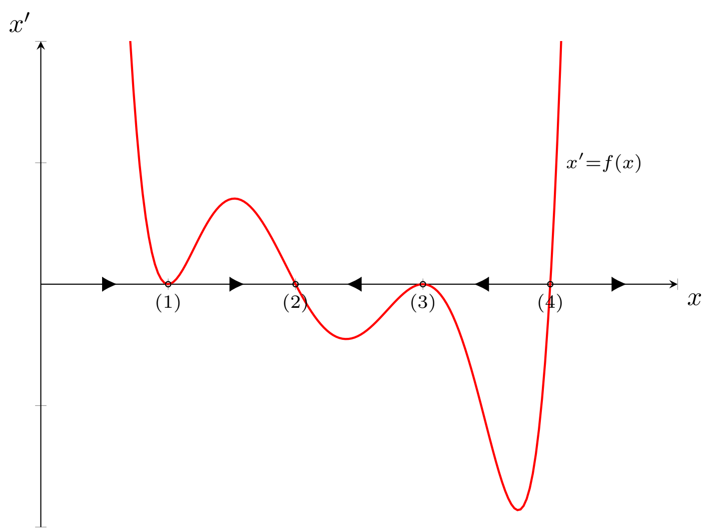
Wherever the graph lies above the line, a particle lying on the x-axis will have positive x', and will therefore move to the right. Similarly, wherever the graph lies below the line, the particle will have a negative x', and will move to the left.

You should indicate these directions with arrows on the x-axis.

If a point where the graph touches the x-axis has arrows pointing inwards, it is *stable*. If it has arrows pointing outwards, it is *unstable*. If arrows point inwards from one direction and outwards from another, it is *structurally unstable*.

The three prior cases are also known collectively as *fixed points*, *stationary points* or *equilibria*.

Example.



In the diagram above, (1) is structurally unstable, (2) is stable, (3) is structurally unstable (but in a different manner than (1)), and (4) is stable.

We have not solved the ODE, but have still managed to determine some qualitative behaviours of the solutions. Notice that a particle that starts past point (4) will move to the right indefinitely, while a particle that starts anywhere to the left will eventually hit point (1). We call the behaviour of a solution as $t \to \infty$ the *asymptotic behaviour* or sometimes the *large time limit*, if the limit is well defined.

The stability of a fixed point clearly depends on how the line interacts with the x-axis. If the line has positive gradient when crossing the x-axis, the point is unstable, and if negative, stable. If the line touches the x-axis, but does not cross it, then the point is structurally unstable.

Note: having a gradient of zero is not sufficient (although necessary) to determine if a fixed point is structurally unstable. For example, the graph of $x' = x^3$ has zero gradient at $x = 0$, but still crosses the x-axis, causing it to be unstable. You should always draw a diagram.

2.4 Euler's Method

Consider the ODE,

$$x' = f(x,t), x(0) = X$$

and suppose we cannot find an analytic solution.

In *Euler's Method*, we find a numerical approximation to the solution.

First, we pick a small time step, h, and assume that x' is approximately constant over the small time step h. With that assumption, we use the Taylor expansion of $x(t + h)$,

$$\begin{aligned} x(t+h) &= x(t) + hx'(t) \\ &= x(t) + hf(x(t),t) \end{aligned}$$

so, the solution to the DE is approximated by the recurrence relation,

$$x(n+1) = x(n) + hf(x(n),nh)$$

Note that we only use the first two terms of the Taylor series, as any further derivatives are 0, since we assume $x'(t)$ is constant.

3 Second Order

3.1 Homogeneous

$$ax'' + bx' + cx = 0$$

Form and solve the *characteristic* or *auxiliary* equation,

$$a\lambda^2 + b\lambda + c = 0$$

There are three cases:

- *Two real roots*: If $\lambda = \alpha,\beta$, then $x = Ae^{\alpha t} + Be^{\beta t}$, where A and B are constant coefficients to be found.
- *Repeated real root*: If $\lambda = \alpha$ with multiplicity 2, then $x = (A + Bt)e^{\alpha t}$.
- *Complex roots*: If $\lambda = p \pm iq$, then $x = e^{pt}(A\cos(qt) + B\sin(qt))$

3.2 Damping

In the above equation, if,

$$\begin{aligned} b = 0,&\ \text{then the system is } \textbf{\textit{un}}\textit{damped};\\ b^2 - 4ac < 0,&\ \text{then the system is } \textbf{\textit{under}}\textit{damped};\\ b^2 - 4ac = 0,&\ \text{then the system is } \textbf{\textit{critically}}\ \textit{damped};\\ b^2 - 4ac > 0,&\ \text{then the system is } \textbf{\textit{over}}\textit{damped.} \end{aligned}$$

An undamped system represents a system without friction, and will oscillate regularly forever. Underdamped systems still oscillate, but a little bit of friction is present, causing the amplitude to decay over time. Critically damped systems generally do not oscillate, simply decaying to zero. Overdamped systems behave similarly, but with a slower decay.

3.3 Non-Homogeneous

$$ax'' + bx' + cx = f(t)$$

Solve the homogenous version, $ax'' + bx' + cx = 0$, to get the *complementary function.*

Now, we want the *particular integral* to deal with the non-homogeneous part. We make an ansatz depending on the form of $f(t)$. If $f(t)$ is a polynomial, we set x equal to a general polynomial of the same degree. If $f(t)$ is exponential, we try the same. If $f(t)$ contains a sine, a cosine or both, we try a linear combination of **both**.

i.e., if $f(t) = 3\cos(5t)$, then we try $x = A\cos(5t) + B\sin(5t)$. Note that we keep the 5's intact, and that we use both sines and cosines, despite $f(t)$ only containing cosine.

Common oversight: Furthermore, if the complementary function matches $f(t)$ in any way, we must multiply our ansatz by t to avoid getting a solution we already have.

Find the first and second derivatives of your ansatz, and substitute into the original equation to solve for any unknown constants.

Remember to add the complementary function to your particular integral afterwards to get the general solution.

If you do not want to use the method above (*Undetermined Coefficients*), there is an alternative method: *Variation of Parameters.*

The method of undetermined coefficients only works when $f(t)$ is polynomial, exponential, (hyperbolic) trigonometric, or a linear combination of the previous.

Variation of parameters is a more powerful technique that works on a wider range of functions, but requires a little more work.

If you would like to learn this method, see § 7.2.

3.4 Resonance

We consider the ODE for a mass/spring system,

$$x'' + cx' + \omega^2 x = F\cos(\Omega t)$$

Where $F\cos(\Omega t)$ is some forcing term.

If the system is underdamped (§ 3.2), this ODE has the solution,

$$x(t) = A\cos(\Omega t - \phi) + Be^{-\frac{ct}{2}}\cos(\alpha t + \delta)$$

for very complicated and mostly irrelevant constants, α, A and ϕ.

But notice how as $t \to \infty$, the second term tends to 0 due to the negative exponential. This second term is the *transient behaviour* term, while the first term is the *steady state solution*.

If there is no forcing and no friction, i.e., $F = 0$ and $c = 0$, $\alpha = \omega$, and the system oscillates as a whole with *natural frequency* $\frac{\omega}{2\pi}$.

If forcing is present, then, as $\Omega \to \omega$, $A \to \infty$. This effect is *resonance*.

4 Recurrence Relations

The methods outlined here I learned outside of this course, so use at your own risk.

4.1 First Order

4.1.1 Homogeneous

Consider the recurrence relation,

$$x_n = ax_{n-1}$$

The solution can be found using *back substitution*:

$$\begin{aligned} x_n &= ax_{n-1} \\ &= a^2x_{n-2} \\ &= a^3x_{n-3} \\ &\ \ \vdots \\ &= a^nx_0 \end{aligned}$$

If initial conditions aren't given, then $x_n = Aa^n$ will suffice.

4.1.2 Non-Homogeneous

$$x_n = ax_{n-1} + f(n)$$

Solve the homogeneous version, $x_n = ax_{n-1}$, to get the complementary solution. Then choose an ansatz using the same procedure as outlined in § 3.3 and substitute it into the non-homogeneous solution to solve for any unknowns.

Note, if $a = 1$ and $f(n)$ is a polynomial, you need to multiply your ansatz by n. But also, if $a = 1$, it may be easier to do back substitution anyway, so keep that in mind.

4.2 Second Order

4.2.1 Homogeneous

Homogeneous

$$ax_{n+2} + bx_{n+1} + cx_n = 0$$

Solve the characteristic equation,

$$a\lambda^2 + b\lambda + c = 0$$

Again, there are three cases:

- *Two real roots*: If $\lambda = \alpha,\beta$, then $x = A\alpha^n + B\beta^n$, where A and B are constant real coefficients to be found.
- *Repeated real root*: If $\lambda = \alpha$ with multiplicity 2, then $x = (A+Bt)\alpha^n$, where A and B are constant real coefficients to be found.
- *Complex roots*: If $\lambda = p \pm iq$, then convert λ to polar form, $p \pm iq = re^{i\theta}$, and $x = r^n(A\cos(n\theta) + B\sin(n\theta))$. Or, if you hate trig like I do, use the same form as for two real roots, and solve for complex A and B.

4.2.2 Non-Homogeneous

See § 4.1.2 and § 3.3. These are done using the exact same procedure.

The only thing to note is, if you have an exponential in your ansatz, you have to be a little careful: if the base of the exponential is equal to one of the roots of the auxiliary equation, multiply by n, like with ODEs. But if the root is repeated, you need to multiply by n^2.

4.3 Other

A *fixed point* of a recurrence relation, $x_n = f(x_{n-1})$ is a value of $x = k$ such that $f(k) = k$. If $|f'(k)| < 1$, then k is a *stable* fixed point. If $|f'(k) > 1|$, then k is an *unstable* fixed point.

5 Systems of Linear First Order ODEs

Much of the theory in this section depends on knowledge from MA106. Many of the methods may seem rather arbitrary and unexplained when first taught, but will make much more sense once you have completed Linear Algebra.

If you need a refresher for linear algebra, or are reading ahead for the year and wish to learn more, I have also written a guide for that module.

5.1 The Jacobian

The *Jacobian matrix* of a function, $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$, denoted $D\mathbf{f}$, is the matrix of partial derivatives,

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} & \cdots & \frac{\partial f_3}{\partial x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \frac{\partial f_n}{\partial x_3} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

This can be more compactly written as,

$$\begin{bmatrix} \frac{\partial \mathbf{f}}{\partial x_1} & \cdots & \frac{\partial \mathbf{f}}{\partial x_n} \end{bmatrix}$$

or,

$$\begin{bmatrix} \nabla^{\mathrm{T}} f_1 \\ \vdots \\ \nabla^{\mathrm{T}} f_n \end{bmatrix}$$

If you are unfamiliar with the ∇ operator, you may look forward to MA134, where you will learn to dread its appearance.

You should familiarise yourself well with the Jacobian, as it appears everywhere in calculus.

Using the Jacobian, we can now define:

5.2 Existence and Uniqueness 2: Electric Boogaloo

$$\frac{d}{dt}(\mathbf{x}(t)) = \mathbf{f}(\mathbf{x},t)$$

If $\mathbf{f}(\mathbf{x},t)$ and $D\mathbf{f}(\mathbf{x},t)$ exist and are continuous for $\mathbf{x} \in U \subseteq \mathbb{R}^n$ and $t \in (a,b)$, then for any $\mathbf{X} \in U$ and $T \in (a,b)$, there exists a unique solution to the equation above on some open interval containing T.

Now, with all the preamble done, we can move onto solving systems of ODEs.

5.3 Homogeneous 2×2 Systems with Constant Coefficients

The system of ODEs,

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned}$$

can be written as a matrix equation,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}'$$

or somewhat less descriptively as,

$$\mathbf{A}\mathbf{x} = \mathbf{x}'$$

Now, find the eigenvalues and eigenvectors of the matrix equation. Again, if you want a more detailed overview on how to actually perform those computations, please read my guide for MA106.

There is a method to solve matrix differential equations that avoids finding the eigenvalues and eigenvectors, but requires possibly more difficult calculations. This method, *matrix exponentiation*, is discussed in § 7.8.

While not particularly feasible to do by hand, it can be easier to use matrix exponentiation than to find eigenvalues and eigenvectors when using a computer, and understanding the method conceptually can be useful.

5.3.1 Distinct Real Eigenvalues

If u,v, $\mathbf{u}$, and $\mathbf{v}$ are distinct eigenvalues and their corresponding eigenvalues of $\mathbf{A}$, then the general solution is given by,

$$\mathbf{x} = Ae^{ut}\mathbf{u} + Be^{vt}\mathbf{v}$$

where A and B are constant coefficients to be determined.

5.3.2 Complex Eigenvalues

If $u = p + iq$ is a complex eigenvalue with corresponding eigenvector, $\mathbf{u} = \begin{bmatrix} a+ib \\ c+id \end{bmatrix}$, then we write,

$$\begin{aligned} \mathbf{x} &= Ae^{ut}\mathbf{u} \\ &= e^{(p+iq)t} \begin{bmatrix} a+ib \\ c+id \end{bmatrix} \end{aligned}$$

Using Euler's formula, we can rewrite this as,

$$\begin{aligned} &= e^{pt}(\cos(qt) + i\sin(qt)) \begin{bmatrix} a+ib \\ c+id \end{bmatrix} \\ &= e^{pt} \left(\begin{bmatrix} a\cos(qt) + ib\cos(qt) \\ c\cos(qt) + id\cos(qt) \end{bmatrix} + \begin{bmatrix} ia\sin(qt) - b\sin(qt) \\ ic\sin(qt) - d\sin(qt) \end{bmatrix} \right) \\ &= e^{pt} \Bigg(\underbrace{\begin{bmatrix} a\cos(qt) - b\sin(qt) \\ c\cos(qt) - d\sin(qt) \end{bmatrix}}_{\mathbf{v}_1(t)} + i \underbrace{\begin{bmatrix} a\sin(qt) + b\cos(qt) \\ c\sin(qt) + d\cos(qt) \end{bmatrix}}_{\mathbf{v}_2(t)} \Bigg) \end{aligned}$$

so we have found two linearly independent solutions, so we can write the general solution as,

$$\mathbf{x} = e^{pt}\left(A\mathbf{v}_1(t) + B\mathbf{v}_2(t)\right)$$

where A and B are constant coefficients to be determined.

Note that you only have to do this process with one eigenvalue and eigenvector, as the other set will differ only by a minus sign, which eventually gets absorbed into the constant coefficients.

5.3.3 Repeated Real Eigenvalues

If λ is a eigenvalue with multiplicity 2, then find a vector, $\mathbf{v_1}$ that satisfies,

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_1 = \mathbf{0}$$

then, a second vector, $\mathbf{v}_2$ that satisfies,

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1$$

The general solution is then given by,

$$e^{\lambda t}(A\mathbf{v}_1 + B(\mathbf{v}_2 + t\mathbf{v}_1))$$

5.4 Diagonalisation and Decoupling

If $\mathbf{Ax} = \mathbf{x}'$, and $\mathbf{A}$ has distinct eigenvalues, swapping to an eigenbasis will let you decouple a system of ODEs by defining a new variable in the eigenbasis.

i.e., Let $\mathbf{x}' = \mathbf{Ax}$, and suppose A has eigenvalues u and v and corresponding eigenvectors $\mathbf{u}$ and $\mathbf{v}$. Let P be the matrix with $\mathbf{u}$ and $\mathbf{v}$ as columns. P is a change of basis matrix from the eigenbasis, Y to the canonical basis, X:

$$\begin{tikzcd} X \arrow[r, "A"] & X' \\ Y \arrow[u, "P"'] \arrow[r, "B"] & Y' \arrow[u, "P"'] \end{tikzcd}$$

We see that $\mathbf{B} = \mathbf{P}^{-1}\mathbf{AP}$. Being in an eigenbasis, $\mathbf{B}$ will be a diagonal matrix with u and v along the diagonals.

Let $\mathbf{W} = \mathbf{P}^{-1}\mathbf{x}$, so,

$$\begin{aligned}
\mathbf{W}' &= \mathbf{P}^{-1}\mathbf{x}' \\
&= \mathbf{P}^{-1}\mathbf{Ax} \\
&= \mathbf{P}^{-1}\mathbf{APW} \\
&= \mathbf{P}^{-1}\mathbf{BW} \\
\begin{bmatrix} w' \\ z' \end{bmatrix} &= \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix}
\end{aligned}$$

So $w' = uw$ and $z' = vz$, so $w = Ce^{ut}$ and $z = De^{vt}$, where C and D are constant coefficients to be found.

Decoupling can also be done without transforming into an eigenbasis by defining new variables in the right way.

Example. Transform the third-order homogeneous differential equation,

$$\frac{d^3x}{dt^3} - 3\frac{dx}{dt} - 2x = 0$$

into a system of three first-order differential equations.

Let $x = x$, $y = x'$ and $z = x''$.

$$\begin{aligned}
\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} x' \\ x'' \\ x''' \end{bmatrix} \\
\frac{dx}{dt} &= x' \\
&= y \\
\frac{dy}{dt} &= x'' \\
&= z \\
\frac{dx}{dt} &= x''' \\
\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \\
x''' - 3x' - 2x &= 0 \\
x''' &= 3x' + 2x \\
x''' &= 3y + 2x \\
\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}
\end{aligned}$$

5.5 Phase Portraits

Find all eigenvalues and eigenvectors of the system.

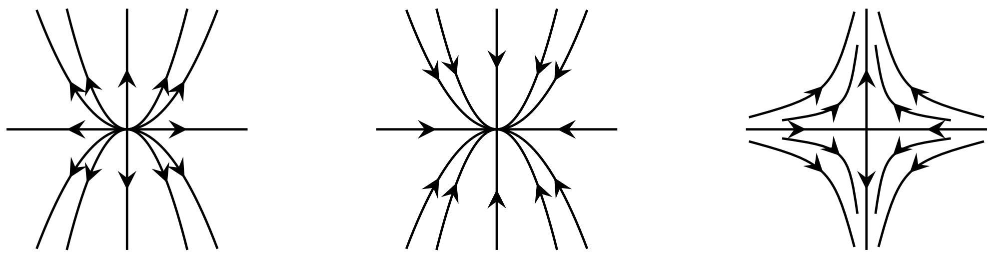
The following sections may be easier to remember if you recall the geometric interpretation of eigenvalues - the real part represents the local scaling, while the imaginary part represents the local rotation.

5.5.1 Distinct Real Eigenvalues

Draw the span of the eigenvectors, with arrows pointing outwards from the origin if the eigenvalue is positive, and inwards if negative.

If your eigenvalues are,

- both positive,
  - If you have eigenvalues, say, 3 and 2, then $e^{3t} \ggg e^{2t}$ as $t \to \infty$, so your trajectories should tend towards being parallel to the eigenvector with eigenvalue 3.
  - This is an *unstable node.*
- both negative,
  - With similar reasoning, your trajectories should tend towards being parallel to the eigenvector with the larger absolute value of eigenvalue.
  - This is a *stable node.*
- one positive, one negative,
  - One line should point inwards, and one points outwards.
  - Draw hyperbolae-esque trajectories between the lines as expected.
  - This is a *saddle point.*



$0 < \lambda_1 < \lambda_2$ Unstable Node

$\lambda_1 < \lambda_2 < 0$ Stable Node

$\lambda_1 < 0 < \lambda_2$ Saddle Point

All three figures have $\mathbf{\hat{\imath}}$ and $\mathbf{\hat{\jmath}}$ as eigenvectors.

5.5.2 Complex Eigenvalues

Say the system

$$\mathbf{A}\mathbf{x} = \mathbf{x}'$$

has matrix

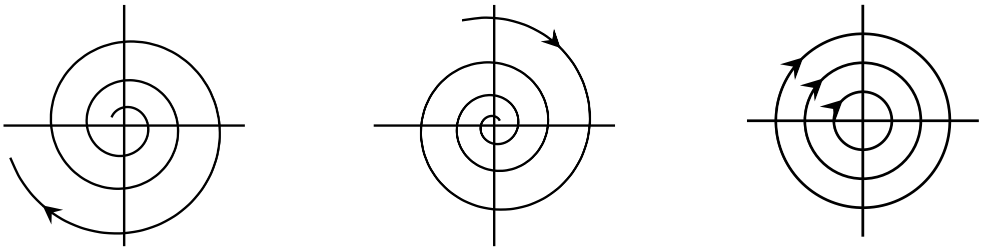
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and you have eigenvalues $p + qi$, with $q \neq 0$. Then, if,

- $p > 0$, the trajectories will spiral outwards, an *unstable spiral*;
- $p < 0$, the trajectories will spiral inwards, a *stable spiral*;

- $p = 0$, i.e., the eigenvalues are purely imaginary, the trajecctories will form circles or ellipses around the origin, a *centre*.

In all three cases, the motion is clockwise if $b - c > 0$, and anticlockwise if $b - c < 0$.



$\Re(\lambda) > 0, \Im(\lambda) \neq 0$
Unstable Spiral

$\Re(\lambda) < 0, \Im(\lambda) \neq 0$
Stable Spiral

$\Re(\lambda) = 0, \Im(\lambda) \neq 0$
Centre

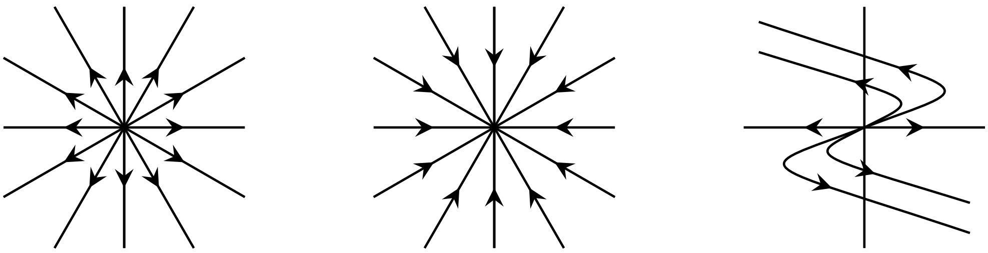
All three figures have $b - c < 0$.

5.5.3 Repeated Real Eigenvalues

If the matrix is a multiple of the identity, then trajectories just point outwards/inwards evenly. This is a *star*, pointing outwards/is stable if the eigenvector is positive and pointing inwards/is unstable if negative. (Geometrically, if it's a multiple of the identity, then it's locally just a scaling transformation, so everything just moves directing in or out from the fixed point).

If the matrix is not a multiple of the identity, then sample some random points to get an idea of what the trajectories should look like. Easy points to sample are (1,0), (0,1), $(-1,0)$ and $(0,-1)$. You can get either an *improper node*, unstable if eigenvalue is positive, stable if negative, or a line of *(un)stable fixed points*. An unstable improper node can also be called a *degenerate sink*, and a stable improper node a *degenerate source*.

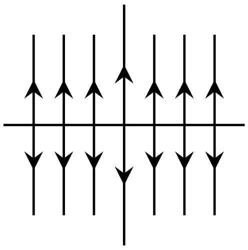
The trajectories in an improper node are parallel to the span of the eigenvector near the origin, then completely reverse in direction. For lines of (un)stable fixed points, the parallel sets of trajectories flow into or out from the line spanned by the eigenvector.



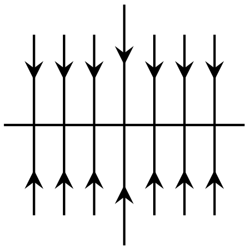
$\Re(\lambda) > 0, \Im(\lambda) \neq 0$
Unstable Star

$\Re(\lambda) < 0, \Im(\lambda) \neq 0$
Stable Star

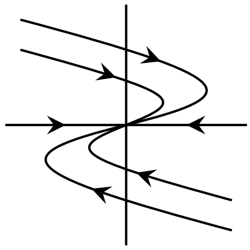
$\Re(\lambda) > 0, \Im(\lambda) \neq 0$
Unstable Improper Node



$\Re(\lambda) > 0, \Im(\lambda) \neq 0$
Line of Unstable Fixed Points



$\Re(\lambda) < 0, \Im(\lambda) \neq 0$
Line of Stable Fixed Points



$\Re(\lambda) < 0, \Im(\lambda) \neq 0$
Stable Improper Node

If you want a more general way to classify all these points, you can compute the trace and determinant of the matrix $\mathbf{A}$.

Let $\Delta = (\operatorname{Tr}\mathbf{A})^2 - 4\det\mathbf{A}$.

$\Delta, \operatorname{Tr}\mathbf{A}, \det\mathbf{A} = 0$, then the matrix is the zero matrix and every point is locally a fixed point;

$\det\mathbf{A} < 0$ - saddle;

$\Delta > 0, \operatorname{Tr}\mathbf{A} > 0, \det\mathbf{A} = 0$ - line of unstable fixed points;

$\Delta > 0, \operatorname{Tr}\mathbf{A} < 0, \det\mathbf{A} = 0$ - line of stable fixed points;

$\Delta > 0, \operatorname{Tr}\mathbf{A} > 0, \det\mathbf{A} > 0$ - unstable node;

$\Delta > 0, \operatorname{Tr}\mathbf{A} < 0, \det\mathbf{A} > 0$ - stable node;

$\Delta = 0, \operatorname{Tr}\mathbf{A} > 0, \det\mathbf{A} > 0$ - unstable improper node;

$\Delta = 0, \operatorname{Tr}\mathbf{A} < 0, \det\mathbf{A} > 0$ - stable improper node;

$\Delta < 0, \operatorname{Tr}\mathbf{A} > 0, \det\mathbf{A} > 0$ - unstable spiral;

$\Delta < 0, \operatorname{Tr}\mathbf{A} < 0, \det\mathbf{A} > 0$ - stable spiral;

$\mathbf{A} = k\mathbf{I}, k > 0$ - unstable star;

$\mathbf{A} = k\mathbf{I}, k < 0$ - stable star;

$\Delta < 0, \operatorname{Tr}\mathbf{A} = 0, \det\mathbf{A} > 0$ - centre.

5.6 Local Linearisation near Fixed Points

If we have the system,

$$x' = f(x,y)$$
$$y' = g(x,y)$$

where f and/or g are non-linear, and you are asked to draw a phase diagram of fixed points of this system, evaluate the Jacobian (§ 5.1) at each fixed point and use it as your matrix for determining eigenvalues/eigenvectors.

Example. Consider the system,

$$x' = y \tag{1}$$
$$y' = -x + y - x^2 y \tag{2}$$

(This is the *Van der Pol oscillator*). Find and classify a fixed point of this system.

Clearly, (0,0) is a fixed point of this system. But what does the phase diagram look like?

First, compute the Jacobian:

$$D\mathbf{X} = \begin{bmatrix} 0 & 1 \\ -1-2xy & 1-x^2 \end{bmatrix}$$

and evaluate it at our fixed point,

$$D\mathbf{X} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

which has eigenvalues $\frac{1}{2} \pm \frac{\sqrt{3}}{2}$, indicating that the phase portrait around the fixed point is locally an unstable spiral.

6 Closing Remarks & Condensed Summary

As was mentioned in the introduction, this module is extremely computational in nature, and doesn't require a lot of conceptual understanding. Do some practice if you haven't integrated or solved recurrence relations in a while, and you'll be fine.

If you memorise just the list below, you'll almost certainly pass.

- DEs:
  - Memorise what linear, homogeneous, autonomous, etc. mean. It's an easy 4 or 5 marks.
  - Your methods of constructing and solving DEs should be the same as from A-Levels, so you should be okay on those.
  - If $\frac{dx}{dt} = f(x,t)$ and $x(a) = b$, then,
    * A solution exists if $f(x,t)$ is continuous near (a,b).
    * The solution is unique if $\frac{\partial f}{\partial x}$ is continuous near (a,b).
  - Fundemental Theorem of Calculus:
    * Suppose $f : [a,b] \to \mathbb{R}$ is continuous.
    * Let $G(x) = \int_a^x f(z)\, dz$.
    * Then, $\frac{d}{dx}G(x) = f(x)$.
    * Furthermore, $\int_a^b f(x)\, dx = F(a) - F(b)$ for any F such that $F'(x) = f(x)$.
- Recurrence relations, you solve in the exact same way as DEs. Just take care to multiply your ansatz by n^2 if it is an exponential with base equal to a repeated root.
- Decoupling: Let $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$ and write down the matrix with eigenvalues on the diagonal.
- Phase Portraits:
  - Distinct & real eigenvalues - draw aigenvectors, and trajectories tend towards the one with larger magnitude (*nodes* and *saddles*).
  - Complex eigenvalues - spirals inwards if $\Re(\lambda) < 0$, outwards if $\Re(\lambda) > 0$ and forms circles or ellipses if $\Re(\lambda) = 0$, clockwise if $b - c > 0$ and anticlockwise if $b - c < 0$, where b and c are from the equation matrix (*spirals* and *centres*).
  - Repeated real eigenvalue - if the equation matrix is a multiple of the identity, it's a *star*. Otherwise, sample random points to test for an *improper node* or a *line of (un)stable points.*
  - If you're ever unsure as to where your trajectory should go, sample some points. $\pm\hat{\mathbf{\imath}}$ and $\pm\hat{\mathbf{\jmath}}$ are often easy to check.
- Linearising - evaluate the Jacobian of your vector function at a fixed point to locally linearise the system.

7 Additional Techniques

This section will cover further techniques for integration that you may find faster and/or easier to perform. These techniques are not examinable, but I highly recommend at least learning tabular integration by parts, as it streamlines the method taught at A-Level to an extreme degree, particularly for repeated applications of integration by parts.

7.1 Tabular Integration by Parts

Say we want to integrate this function,

$$\int a(x)b(x)\,dx$$

Being a product of two functions, we use integration by parts.

Draw out a table, with D above the first column and I above the first, then put a column of alternating plusses and minuses, besides the first, starting with a plus. You'll get a feel for how many rows you'll need as you get more used to using this method, but for now, I've drawn 4.

	D	I
$+$		
$-$		
$+$		
$-$		

Now, look at the integral, and decide which function is easier to differentiate. Or more usually, which function you don't want to integrate. Suppose we don't want to integrate $a(x)$, so we differentiate $a(x)$ and integrate $b(x)$.

Put $a(x)$ under D, and $b(x)$ under I, and differentiate and integrate them repeatedly, putting the result in the next row each time. For ease of reading, let $b_{.}(x)$ indicate the first integral of $b(x)$, $b_{..}(x)$ the second, and so on.

	D	I
$+$	$a(x)$	$b(x)$
$-$	$a'(x)$	$b_{.}(x)$
$+$	$a''(x)$	$b_{..}(x)$
$-$	$a'''(x)$	$b_{...}(x)$

When we decide to stop (I've done 3 additional rows here), multiply diagonal elements, keeping the signs attached. Then, multiply the final row horizontally and throw it into an integral;

	D		I
$+$	$a(x)$	$\times$	$b(x)$
$-$	$a'(x)$	$\times$	$b_{.}(x)$
$+$	$a''(x)$	$\times$	$b_{..}(x)$
$-$	$a'''(x)$		$b_{...}(x)$

$$\int a(x)b(x)\,dx = [+a(x)b_{.}(x)] + [-a'(x)b_{..}(x)] + [a''(x)b_{...}(x)] + \int [-a'''(x)b_{...}(x)]\,dx$$

But when do we know when to stop?

There are three main stops:

- There is a 0 in the D column.
- You can integrate a row.
- A row appears more than once.

In the first case, when you multiply the last row together, the final integral just disappears. In the second case, if you can integrate a row, just stop the process and do the integral. In the third case, if a row appears more than once, that means you can rewrite the original integral in terms of itself, plus some extra stuff at the front, which you can rearrange for.

Example. Evaluate,

$$\int x^3 \sin(4x)\,dx$$

It's almost always ideal to differentiate the polynomial, as we know we can eventually get it to 0. The sine function is fine to integrate as well, so let's do that.

	D		I
$+$	x^3	$\times$	$\sin(4x)$
$-$	$3x^2$	$\times$	$-\frac{1}{4}\cos(4x)$
$+$	$6x$	$\times$	$-\frac{1}{16}\sin(4x)$
$-$	6	$\times$	$\frac{1}{64}\cos(4x)$
$+$	0		$\frac{1}{256}\sin(4x)$

$$\int x^3 \sin(4x)\,dx = -\frac{1}{4}x^3\cos(4x) + \frac{3}{16}x^2\sin(4x) + \frac{3}{32}x\cos(4x) - \frac{3}{128}\sin(4x)$$

Example. Evaluate,

$$\int x^3 \ln x\,dx$$

We like to differentiate polynomials, but integrating $\ln x$ requires integration by parts in the first place, which we would like to avoid, especially if we are repeatedly integrating it. So, we differentiate $\ln x$ and integrate x^3.

	D	I
$+$	$\ln x$	x^3
$-$	$\frac{1}{x}$	$\frac{1}{4}x^4$

If we look at the final row, we can already integrate its product, so we stop.

	D	I
$+$	$\ln x$	x^3
$-$	$\frac{1}{x}$	$\frac{1}{4}x^4$

$$\begin{aligned}\int x^3 \ln x \, dx &= \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3 \, dx \\ &= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4\end{aligned}$$

Example. Evaluate,

$$\int e^x \sin x \, dx$$

e^x and $\sin x$ are both easy to integrate and differentiate, so it doesn't really matter which way around we put them. Let's differentiate e^x and integrate $\sin x$.

	D	I
$+$	e^x	$\sin(x)$
$-$	e^x	$-\cos(x)$
$+$	e^x	$-\sin(x)$

We see that the final row is a copy of the first one (ignoring signs), so we can rewrite the integral as,

$$\begin{aligned}\int e^x \sin x \, dx &= -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx \\ 2\int e^x \sin x \, dx &= -e^x \cos x + e^x \sin x \\ \int e^x \sin x \, dx &= \frac{1}{2}e^x \sin x - \frac{1}{2}e^x \cos x\end{aligned}$$

7.2 Variation of Parameters

Variation of parameters is a general method to solve non-homogeneous linear ODEs, though it can also be extended to solve PDEs as well.

Here, we will only consider second order ODEs,

$$x'' + bx' + cx = f(t)$$

(we divide through by the constant coefficient of x'' to simplify this method).

Consider the solution to the homogeneous case, which depends on the solutions to the auxiliary equation,

$$\begin{aligned}x &= Ae^{\alpha t} + Be^{\beta t} \\ x &= (A + Bt)e^{\alpha t} \\ x &= e^{pt}(A\cos(qt) + B\sin(qt))\end{aligned}$$

Notice how each solution can be split into two linearly independent parts (see MA106 if you are unfamiliar with linear independence), x_1 and x_2, where,

$$\begin{aligned} x_1 &= Ae^{\alpha t}, & x_2 &= Be^{\beta t} \\ x_1 &= Ae^{\alpha t}, & x_2 &= Bte^{\beta t} \\ x_1 &= Ae^{pt}\cos(qt), & x_2 &= Be^{pt}\sin(qt) \end{aligned}$$

The functions x_1 and x_2 are the *fundamental solutions* of the equation.

We define the *Wronskian matrix* as,

$$\begin{bmatrix} x_1 & x_2 \\ x_1' & x_2' \end{bmatrix}$$

From linear algebra, we know that the *Wronskian determinant*, W, of this matrix cannot be 0. We use the Wronskian determinant to find the particular integral of the equation.

$$x = -x_1 \int \frac{x_2 f}{W}\, dx + x_2 \int \frac{x_1 f}{W}\, dx$$

Remember to add the complementary function to your particular integral afterwards to get the general solution.

7.3 Weierstrass Substitution

The *Weierstrass substitution* is a change of variable that transforms rational functions of trigonometric functions into an ordinary rational function of a parameter, t.

Letting $t = \tan \frac{x}{2}$, we can transform the integral,

$$\int f(\sin x, \cos x)\, dx = \int f\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right) \frac{2}{1+t^2}\, dt$$

Geometrically, as x varies, the point $(\cos x, \sin x)$ travels across the unit circle at unit speed. In other words, it is a *unit speed parametrisation* (see MA134). The Weierstrass substitution is an alternative parametrisation of the unit circle such that the point $\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$ travels around the unit circle only once as t varies from $-\infty$ to ∞, starting and ending at $(-1{,}0)$. If you are familiar with projective geometry, this substitution can be viewed as the sterographic projection of the unit circle onto the y-axis from the point $(-1{,}0)$. This view can help you rederive various formulae on the fly, if required.

7.4 Reduction Formulae

A *reduction formula* allows you to write a recurrence relation for an integral in terms of related integrals with hopefully smaller exponents.

We do this by splitting up the exponent, substituting if needed, then integrating by parts.

Example.

$$\int \sin^n x\, dx$$

We wish to find a reduction formula for this integral. Start by setting,

$$\begin{aligned} I_n &= \int \sin^n x\, dx \\ &= \int \sin^{n-1} x \sin x\, dx \end{aligned}$$

$$
\begin{aligned}
&= -\sin^{n-1} x\cos x + \int (n-1)\sin^{n-2} x\cos^2 x\,dx \\
&= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)\,dx \\
&= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^2 x\,dx \\
&= -\sin^{n-1} x\cos x + (n-1)I_{n-2} - (n-1)I_n \\
I_n + (n-1)I_n &= -\sin^{n-1} x\cos x + (n-1)I_{n-2} \\
I_n &= -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}I_{n-2}
\end{aligned}
$$

So now, if we're given, for example, $\int \sin^{100} x\,dx$, we can repeatedly apply the reduction formula until the power is low enough for us to evaluate the integral by hand.

7.5 Euler Substitution

If $f(a,b)$ is a rational function, then

$$\int f(x,\sqrt{ax^2+bx+c})\,dx$$

can be changed into the integral of a rational function using *Euler substitutions*.

If $a > 0$, solve $\sqrt{ax^2+bx+c} = \pm x\sqrt{a} + t$ for x (the positive or negative sign can be chosen at will, depending on which is easier). The result will be a rational expression, that also allows us to write dx as a rational expression of t when we perform the substitution.

If $c > 0$, solve $\sqrt{ax^2+bc+c} = xt \pm \sqrt{c}$ for x, and use the result as your substitution. Again, the positive and negative sign can be chosen at will.

If ax^2+bx+c has real roots, α,β, then we solve $\sqrt{a(x-\alpha)(x-\beta)} = (x-\alpha)t$ for x, which will again result in a rational expression.

7.6 Laplace Transformations

This technique is completely overkill for this module, but it can be a very good shortcut if you prefer memorisation based methods.

The *Laplace transform* is an integral transform that converts a real-valued function (often, t) into a complex-valued function (often of a complex variable, s). This transform is useful because linear differential equations transform into simple algebraic equations.

The Laplace transform of a function, $f(t)$, is given by

$$\mathscr{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$$

While this looks intimidating, in practice, you just memorise the transforms of common functions. A short table of such transforms is included below. $\mathscr{L}(f(t))$ is also often written as $F(s)$.

$f(t)$	$\mathscr{L}(f(t))$
c	$\frac{c}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
$t^n e^{-\alpha t}$	$\frac{n!}{(s+\alpha)^{n+1}}$
$e^{-\alpha t}$	$\frac{1}{s+\alpha}$
$1 - e^{-\alpha t}$	$\frac{\alpha}{s(s+\alpha)}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$

In general, multiplying a function by $e^{-\alpha t}$ shifts the s along by α in the transform, i.e., $\mathscr{L}(e^{-\alpha t} f(t)) = F(s+\alpha)$.

If you intend on using Laplace transforms, you should commit this table, and more, to memory, as you will also need to be able to recognise them quickly in order to find the inverse Laplace transform of some given $F(s)$.

Example. Given,

$$F(s) = \frac{s+3}{s^2+6s+13}$$

what is $\mathscr{L}^{-1}(F(s))$?

Completing the square on the denominator, we have $\frac{s+3}{(s+3)^2+4}$, which matches the form for cosine. But the s is shifted along by 3, so we have $f(t) = e^{-3t} \cos 2t$.

An important property of the Laplace transform, is that it is a linear operator (see MA106). We should also look at the effect of taking the Laplace transform of a derivative:

$$\begin{aligned}
\mathscr{L}(f'(t)) &= \int_0^\infty e^{-st} f'(t)\, dt \\
&= e^{-st} f(t)\Big|_0^\infty + \int_0^\infty s e^{-st} f(t)\, dt \\
&= e^{-st} f(t)\Big|_0^\infty + s \int_0^\infty e^{-st} f(t)\, dt \\
&= e^{-st} f(t)\Big|_0^\infty + s\mathscr{L} f(t) \\
&= [0] - [f(0)] + s\mathscr{L} f(t) \\
&= s\mathscr{L}(f(t)) - f(0)
\end{aligned}$$

so we can rewrite the Laplace transform of a derivative as the Laplace transform of the original function, plus an initial condition. Similarly, we have,

$$\begin{aligned}
\mathscr{L}(f''(t)) &= s\mathscr{L}(f'(t)) - f'(0) \\
&= s^2\mathscr{L}(f(t)) - sf(0) - f'(0)
\end{aligned}$$

and this pattern continues for higher derivatives.

Now, let's use the Laplace transform to solve an initial value problem.

Example.

$$x'' + 5x' + 6x = 0, x(0) = 2, x'(0) = 3$$

$$x'' + 5x' + 6x = 0$$

$$\mathscr{L}(x'' + 5x' + 6x) = \mathscr{L}(0)$$

Recall that the Laplace transform is linear, and so,

$$\begin{aligned}
\mathscr{L}(x'') + 5\mathscr{L}(x') + 6\mathscr{L}(x) &= \mathscr{L}(0) \\
(s^2\mathscr{L}(x) - sx(0) - x'(0)) + 5(s\mathscr{L}(x) - x(0)) + 6\mathscr{L}(x) &= 0 \\
(s^2 + 5s + 6)\mathscr{L}(x) &= (s + 5)x(0) + x'(0)
\end{aligned}$$

Use our initial conditions,

$$\begin{aligned}
(s^2 + 5s + 6)\mathscr{L}(x) &= 2s + 13 \\
\mathscr{L}(x) &= \frac{2s + 13}{s^2 + 5s + 6} \\
\mathscr{L}(x) &= \frac{2s + 13}{(s + 2)(s + 3)}
\end{aligned}$$

Performing partial fraction decomposition,

$$\begin{aligned}
\mathscr{L}(x) &= 9\left(\frac{1}{s + 2}\right) - 7\left(\frac{1}{s + 3}\right) \\
\mathscr{L}(x) &= 9\mathscr{L}(e^{-2t}) - 7\mathscr{L}(e^{-3t}) \\
\mathscr{L}(x) &= \mathscr{L}(9e^{-2t} - 7e^{-3t}) \\
x &= 9e^{-2t} - 7e^{-3t}
\end{aligned}$$

It is important to note that you generally cannot find the Laplace transform of the product or composition of two functions. However, due to linearity, as long as your function can be written as the *sum* of known functions, you can work out its Laplace transform.

If you take probability or any kind of electrical engineering or signal/image processing, you may be familiar with *convolution*. You'll be happy to know that the Laplace transform of a convolution is simply the product of the Laplace transforms. That is, $\mathscr{L}((f * g)(t)) = \mathscr{L}(f(t)) \cdot \mathscr{L}(g(t)) = F(s) \cdot G(s)$.

For image processing, which I am most familiar with, convolving an image with a kernel is required for a multitude of operations, including blurring, sharpening and edge detection. But convolving the naive way can be an extremely slow process, especially for large kernels. Many modern convolution functions take an integral transform (often Fourier, rather than Laplace), allowing convolution to be applied as a multiplication, which is much faster to compute, before transforming back to the original image.

7.7 Leibniz Integration Rule

$$\frac{d}{dx}\left(\int_a^b f(x,t)\,dt\right) = \int_a^b \frac{\partial}{\partial x} f(x,t)\,dt$$

There is a longer form for non-constant bounds of integration, but we will focus on the special case of constant bounds.

This theorem allows us to integrate functions we otherwise wouldn't be able to.

Example. Evaluate

$$\int_0^\infty \frac{\sin t}{t}\,dt$$

(The integrand is also known as the (unnormalised) *sinc function*, a function occuring often in signal processing contexts. This particular definite integral is the *Dirichlet integral*, and cannot be evaluated using standard methods.)

We begin by defining a function,

$$f(s) = \int_0^\infty e^{-st} \frac{\sin t}{t} \, dt$$

(The similarity with the earlier Laplace transform is not a coincidence. There is a much faster way of doing this using the Laplace transform, combined with *Abel's theorem*, but that method will not be covered here, as it is beyond the scope of this document.)

We note that $f(0)$ is equal to the desired integral.

$$\begin{aligned}
\frac{df}{ds} &= \frac{d}{ds} \int_0^\infty e^{-st} \frac{\sin t}{t} \, dt \\
&= \int_0^\infty \frac{\partial}{\partial s} e^{-st} \frac{\sin t}{t} \, dt \\
&= -\int_0^\infty e^{-st} \sin t \, dt
\end{aligned}$$

You can alternatively use the complex definition of sine to perform this integral as an exercise.

$$\begin{aligned}
&= -\frac{e^{-st}(\cos t + s \sin t)}{s^2+1} \Bigg|_{t=0}^{t=\infty} \\
&= -\frac{1}{s^2+1}
\end{aligned}$$

Now, we integrate both sides with respect to s.

$$\begin{aligned}
f(s) &= -\int \frac{1}{s^2+1} \, ds \\
&= -\arctan s + C \\
-\arctan s + C &= \int_0^\infty e^{-st} \frac{\sin t}{t} \, dt
\end{aligned}$$

Here, we can try some values of s to get some information about C. $s = 0$ doesn't work, because we just get the original problem back. Let's see what happens as $s \to \infty$.

$$\begin{aligned}
\lim_{s\to\infty} -\arctan s + C &= \lim_{s\to\infty} \int_0^\infty e^{-st} \frac{\sin t}{t} \, dt \\
\lim_{s\to\infty} -\arctan s + C &= \lim_{s\to\infty} \int_0^\infty \frac{\sin t}{te^{st}} \, dt \\
-\frac{\pi}{2} + C &= \int_0^\infty 0 \, dt \\
-\frac{\pi}{2} + C &= 0 \\
C &= \frac{\pi}{2} \\
f(s) &= \frac{\pi}{2} - \arctan s \\
f(0) &= \frac{\pi}{2} - 0 \\
\int_0^\infty e^{-0t} \frac{\sin t}{t} \, dt &= \frac{\pi}{2} \\
\int_0^\infty \frac{\sin t}{t} \, dt &= \frac{\pi}{2}
\end{aligned}$$

7.8 Matrix Exponentiation

This is an extremely brief overview of matrix exponentiation, meant as a preview for content of future modules, rather than as a practical method. Feel free to skip this section if uninterested.

Consider the first order ODE,

$$x' = ax$$

so the rate of change of x is proportional to its size at every given value of t.

Now consider the function, e^{at}. $\frac{d}{dt}e^{at} = ae^{at}$, so we see that this proportionality is the same thing as exponential growth, and indeed the general solution, is given by $x = x_0e^{at}$.

But one way to think about how this solution actually works, is that e^{at} isn't a "solution", but instead is some real-valued function acting on some initial condition determined by x_0, to give how the ODE would behave under those specific conditions.

Now, we can generalise to higher dimensions.

Consider the linear system of equations given by,

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

We can also write the solution as some exponential term acting on an initial condition, just like with the one-dimensional case. But here, the exponential term, rather than being a real-valued function of timme, is a matrix-valued function of time, and the initial condition is a vector instead.

The general solution is given by,

$$\mathbf{x} = \exp(\mathbf{A}t)\mathbf{x}_0$$

Obviously, evaluating an infinite power series of matrices is not very practical to do by hand, but there are ways to simplify such expressions. You will learn more on this in second year.

7.9 Non-Elementary Integrals

If you somehow end up with one of these when constructing a differential equation for a question, you've probably done something wrong earlier.

The following is a non-exhaustive list of integrals that you will not be able to evaluate.

$$\int \sqrt{1+x^n}\,dx, \quad n \in \mathbb{N}, n \geq 3 \qquad \int \sin(\sin x)\,dx \qquad \int e^{e^x}\,dx$$

$$\int \sqrt{1-x^n}\,dx, \quad n \in \mathbb{N}, n \geq 3 \qquad \int \arcsin(\arcsin x)\,dx \qquad \int e^{x^2}\,dx$$

$$\int x^x\,dx \qquad \int \sin(x^2)\,dx \qquad \int e^{-x^2}\,dx$$

$$\int x^{-x}\,dx \qquad \int \cos(x^2)\,dx \qquad \int \frac{e^x}{x}\,dx$$

$$\int \frac{1}{\ln x}\,dx \qquad \int \frac{\sin x}{x}\,dx \qquad \int \frac{e^{-x}}{x}\,dx$$

$$\int \frac{x^n}{e^x-1}\,dx \quad n \in \mathbb{N} \qquad \int \ln(\ln x)\,dx \qquad \int x^{c-1}e^{-x}\,dx, \quad c \notin \mathbb{N}$$

While you don't have to memorise all of these, it's good to be able to recognise when you have an integral you can't evaluate, so you can go back and check your previous working, rather than wasting time on the integral.